ON CONSTRUCTION OF THREE SYMBOL PARTIALLY BALANCED ARRAYS OF STRENGTH TWO

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INTRODUCTION

In a factorial set up of n factors each at s levels there are altogether s^n assemblics of which a sub-set of N assemblies is known as an array. An array where all the s^d assemblies corresponding to any d factors chosen out of the n factors occur an equal number of times has been termed by Rao [6] as hypercube of strength d. Later, Rao [7] extended the definition of hypercube of strength d to cover a wider class of arrays called orthogonal arrays.

Chakravarti [1] introduced still wider a class of arrays which he called partially balanced (*PB*) arrays. An $n \times N$ matrix A with entries from a set Σ of s $(\geqslant 2)$ elements is said to be a partially balanced array with s symbols, n constraints, N assemblies and strength t if every $t \times N$ submatrix of A contains the ordered $t \times 1$ column vector $(x_1, x_2, \ldots\ldots, x_t)$, $x_i \in \Sigma$, $\lambda(x_1, x_2, \ldots\ldots x_t)$ times, where $\lambda(x_1, x_2, \ldots\ldots x_t)$ is a non negative integer and is invariant under any permutation of $x_1, \ldots\ldots, x_t$.

Chakravarti [1] observed that *PB* arrays can be constructed from orthogonal arrays by suitably omitting certain assemblies. He had also shown that main effects of the factors can be estimated from such arrays, and these, therefore can be considered as main effect plans. Chakravarti [2] gave some methods of construction of *PB* arrays. Two symbol *PB* arrays of strength four were constructed by Srivastava and Chopra [8]. Subsequently, Dey *et. al.*, [3] put forward two methods of construction of partially balanced arrays of strength two and three.

This paper presents a method of construction of three symbol *PB* arrays of strength two for v or less factors in $2v$ assemblies when v is a prime ($\neq 2$) or an odd prime power. It may not be out of place to mention here that these *PB* arrays have been used by Gupta [4] and Gupta and Dey [5] to construct main effect plans for 3^n factorials and second order rotatable designs of six levels.

2. The Method Of Construction Of PB Arrays

The method of construction to be described in this section gives us three symbol partially balanced (*PB*) arrays of strength two for v factors in $2v$ assemblies, v being any odd prime or power of prime.

Consider $GF(v)$ where v is any odd prime or prime power. Let x denote a primitive element of the field. Then it is well known that all the v elements of $GF(v)$ can be represented by

$$0,\ x^0,\ x^1,\ x^2,\ \ldots,\ x^{v-2}$$

Let C denote a $(v \times 1)$ row vector $(c_1, c_2, \ldots, c_v)$ consisting of all the v elements of $GF(v)$ each occurring exactly once in some order whatsoever. We define

$$C_a = C + aJ,\ mod\ v \qquad \ldots(1)$$

where a is any element of $GF(v)$ and J is a $(v \times 1)$ row vector of 1's. Now we construct a $(v \times v)$ matrix B the rows of which are given by

$$C_a,\ a \in GF(v).$$

Replacing 0's, even powers of x and odd powers of x in B by 1's, 2's and 0's, respectively we get a $(v \times v)$ matrix A_1, say. Similarly we obtain another matrix A_2 from B by replacing 0's, even powers of x and odd powers of x by 1's, 0's and 2's, respectively.

Then the $(v \times 2v)$ matrix

$$A = [A_1 \ \vdots\ A_2] \qquad \ldots(2)$$

forms, when the columns are treated as assemblies, the required *PB* array of strength two for v factors, in three symbols 0, 1 and 2. In fact if λ_{ij} stands for the number of times the ordered symbol pair (i, j) occurs in the $2v$ ordered pair of symbols that can be formed from any two rows of $A (i, j = 0, 1, 2)$, then we have, irrespective of rows taken

$$\lambda_{00} = (v-3)/2, \quad \lambda_{01} = \lambda_{10} = 1, \quad \lambda_{02} = \lambda_{20} = (v-1)/2,$$
$$\lambda_{11} = 0 \quad , \quad \lambda_{12} = \lambda_{21} = 1, \quad \lambda_{22} = (v-3)/2.$$

We now give a proof of the above statement.

Let $F^1_{ij}(p, q)$ denote the number of times the symbol pair (i, j) occurs among the v ordered pairs of symbols that can be formed from the pth and qth rows of A_1;

$$i, j = 0, 1, 2\ ;\ p, q = 0, 1, 2, \ldots, (v-1);\ p \neq q.$$

Let similar meaning be attached to $F^2_{ij}(p, q)$ when rows are taken from A_2. To prove that the matrix A defined above gives the *PB* array, we have to show that $[F^1_{ij}(p, q) + F^2_{ij}(p, q)]$ is a constant independent of p and q and is dependent only on i, j. We shall show below that this constant is equal to λ_{ij}, expressions for which are given above.

If we denote the zero, odd powers of x and even powers of x of $GF(v)$, by Z, 0 and E respectively, then any pair of elements formed from any two rows of B can be characterised by one of the nine pairs (Z, Z), $(Z, 0)$, (Z, E), $(0, Z)$, $(0, 0)$, $(0, E)$, (E, Z), $(E, 0)$, and (E, E). This is also to be noted that these pairs transform respectively to the symbol pairs (1, 1), (1, 0), (1, 2), (0, 1), (0, 0), (0, 2), (2, 1), (2, 0) and (2, 2) in A_1 and to (1, 1), (1, 2), (1, 0), (2, 1), (2, 2), (2, 0), (0, 1), (0, 2) and (0, 0) in A_2.

Thus if $N_{ij}(p, q)$ stands for the frequencies of pairs in B just as $F^1_{ij}(p, q)$ do in A_1 etc., with $i, j = Z$, 0, and E in $N_{ij}(p, q)$, then we have :

$$
\begin{aligned}
F^1_{00}(p, q) + F^2_{00}(p, q) &= N_{00}(p, q) + N_{EE}(p, q)\\
F^1_{01}(p, q) + F^2_{01}(p, q) &= N_{0Z}(p, q) + N_{EZ}(p, q)\\
F^1_{02}(p, q) + F^2_{02}(p, q) &= N_{0E}(p, q) + N_{E0}(p, q)\\
F^1_{10}(p, q) + F^2_{10}(p, q) &= N_{Z0}(p, q) + N_{ZE}(p, q)\\
F^1_{11}(p, q) + F^2_{11}(p, q) &= N_{ZZ}(p, q) + N_{ZZ}(p, q) \qquad \ldots(3)\\
F^1_{12}(p, q) + F^2_{12}(p, q) &= N_{ZE}(p, q) + N_{Z0}(p, q)\\
F^1_{20}(p, q) + F^2_{20}(p, q) &= N_{E0}(p, q) + N_{0E}(p, q)\\
F^1_{21}(p, q) + F^2_{21}(p, q) &= N_{EZ}(p, q) + N_{0Z}(p, q)\\
F^1_{22}(p, q) + F^2_{22}(p, q) &= N_{EE}(p, q) + N_{00}(p, q)
\end{aligned}
$$

Let, now, without loss of generality the pth and qth rows of B be C_{a_p} and C_{a_q} $(a_p \neq a_q)$ as defined in (1). We note that C_{a_q} can be written as $C_{a_p} + dJ$ where $d = (a_q - a_p)$ and that C_{a_p} is simply a $(v \times 1)$ row vector of all the v elements of $GF(v)$ each occurring exactly once in some order.

We now define

$$T_{ij}(d) = \text{Number of } j\text{'s in } S_i,\ i, j = Z, E, 0; \qquad \ldots(4)$$

where

$$
\begin{aligned}
S_Z &= (0 + d),\\
S_E &= [(x^0 + d), (x^2 + d), \ldots\ldots, (x^{v-3} + d)]\\
S_0 &= [(x^1 + d), (x^3 + d), \ldots\ldots, (x^{v-2} + d)]
\end{aligned}
$$

additions being done mod v.

Then, we clearly have

$$N_{ij}(p, q)=T_{ij}(d)\ ;\ i, j=Z, E, 0 \qquad \ldots(5)$$

To complete the proof, we have now to determine the expressions for the T quantities. The following four distinct cases may arise :

Case (i): d is an even power of x and v is of the form $4n+3(n>0)$.
Case (ii): d is an odd power of x and v is of the form $4n+3$.
Case (iii): d is an even power of x and v is of the form $4n+1$.
Case (iv): d is an odd power of x and v is of the form $4n+1$.

We deal with these cases separately.

Case (i) $v=4n+3$, *d even power of x.*

Since d is an even power of x, let us consider $d=x^{2a}$.

Now x is a primitive element of $GF(v)$; hence $x^{(v-1)/2}=-1$.

So, $x^{2a+(v-1)/2}=x^{p_0}=-d$. $\qquad \ldots(6)$

As $v=4n+3$, we can see p_0 is an odd integer and $x^{2a+(v-1)/2}+d=0$ is an element of S_0.

Again $\frac{d}{x}$, $\frac{d}{x^3}$,......, $\frac{d}{x^{v-2}}$ are all distinct elements and odd powers of x. So

$$\begin{aligned} \frac{d}{x}(x+d)&=\left(\frac{d^2}{x}+d\right)\\ \frac{d}{x^3}(x^3+d)&=\left(\frac{d^2}{x^3}+d\right)\\ &\vdots \\ \frac{d}{x^{v-2}}(x^{v-2}+d)&=\left(\frac{d^2}{x^{v-2}}+d\right) \end{aligned} \qquad \ldots(7)$$

Now, $\frac{d^2}{x}$, $\frac{d^2}{x^3}$,......, $\frac{d^2}{x^{v-2}}$ are all distinct elements of $GF(v)$ and are odd powers of x.

So $\left(\frac{d^2}{x}+d\right)$, $\left(\frac{d^2}{x^3}+d\right)$,....., $\left(\frac{d^2}{x^{v-2}}+d\right)$ are also elements of S_0.

Thus we can see except one element which is zero, all other $(v-3)/2$ elements of S_0 can be grouped into $(v-3)/4$ pairs such that one member of every pair is equal to the other member of it multiplied by some odd power of x. This shows that there are exactly $(v-3)/4$ even powers of x and $(v-3)/4$ odd powers of x in S_0.

Now let us consider the combined set

$$S=(S_E, S_0). \qquad \ldots(8)$$

We observe that all the even and odd powers of x except one even power (d itself) appear in S. Since there are $(v-1)/2$ even and $(v-1)/2$ odd powers of x in $GF(v)$ it is clear that S_E contains $(V-3)/4$ even powers of x and $(V+1)/4$ odd powers of x. Obviously there cannot be any zero in S_E. The only element of S_Z is an even power of x. Hence we get the following table of frequencies.

TABLE 1

v=4n+3, d=even power of x
Values of $T_{ij}(d)$

i \ j	Z	E	0
Z	0	1	0
E	0	$(v-3)/4$	$(v+1)/4$
0	1	$(v-3)/4$	$(v-3)/4$

Case (ii) $v=4n+3$, d=*odd power of* x

When d is an odd power of x and v is of the form $4n+3$, S_E contains one zero and S_0 does not contain any zero. This is also to note here : (*i*) S_E of case (*ii*) consists of elements of S_0 of case (*i*) each multiplied by some odd power of x; and (*ii*) S_0 of case (*ii*) consists of elements of S_E of case (*i*) each multiplied by some odd power of x.

Thus the results of case (*ii*) are immediately available from those of case (*i*) and is presented in the following table.

TABLE 2

v=4n+3 d=odd power of x
Values of $T_{ij}(d)$

i \ j	Z	E	0
Z	0	0	1
E	1	$(v-3)/4$	$(v-3)/4$
0	0	$(v+1)/4$	$(v-3)/4$

Case (iii) and Case (iv) :

Following the same lines as above we can arrive at the frequencies $T_{ij}(d)$ shown in the following tables.

TABLE 3

$v=4n+1$ d=even power of x
Values of $T_{ij}(d)$

j	Z	E	0
Z	0	1	0
E	1	$(v-5)/4$	$(v-1)/4$
0	0	$(v-1)/4$	$(v-1)/4$

TABLE 4

$v=4n+1$, d=odd power of x
Values of $T_{ij}(d)$

j	Z	E	0
Z	0	0	1
E	0	$(v-1)/4$	$(v-1)/4$
0	1	$(v-1)/4$	$(v-5)/4$

It is now easy to see that the equations at (3) and (5) along with the tabular frequencies given above, when combined together, give the following λ_{ij}'s, the frequencies of the symbol pairs (i, j), $i, j=0, 1, 2$, occurring in the *PB* array :

$$\lambda_{00}=(v-3)/2,$$
$$\lambda_{01}=\lambda_{10}=1,$$
$$\lambda_{02}=\lambda_{20}=(v-1)/2,$$
$$\lambda_{11}=0,$$
$$\lambda_{12}=\lambda_{21}=1,$$
$$\lambda_{22}=(v-3)/2.$$

3. An Example

Let us construct a three symbol *PB* array for $3^2=9$ factors in 18 assemblies. The elements of $GF(3^2)$ are $(0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2)$, and reductions are done mod $[3, P(x)]$ where $P(x)$, a minimum, polynomial, is x^2+x+2. One can easily see that : $x^0=1$, $x^1=x$, $x^2=2x+1$, $x^3=2x+2$, $x^4=2$, $x^5=2x$, $x^6=x+2$, $x^7=x+1$.

Let

$$B=\begin{bmatrix} 0 & 1 & x & 2x+1 & 2x+2 & 2 & 2x & x+2 & x+1 \\ 1 & 2 & x+1 & 2x+2 & 2x & 0 & 2x+1 & x & x+2 \\ x & x+1 & 2x & 1 & 2 & x+2 & 0 & 2x+2 & 2x+1 \\ 2x+1 & 2x+2 & 1 & x+2 & x & 2x & x+1 & 0 & 2 \\ 2x+2 & 2x & 2 & x & x+1 & 2x+1 & x+2 & 1 & 0 \\ 2 & 0 & x+2 & 2x & 2x+1 & 1 & 2x+2 & x+1 & x \\ 2x & 2x+1 & 0 & x+1 & x+2 & 2x+2 & x & 2 & 1 \\ x+2 & x & 2x+2 & 0 & 1 & x+1 & 2 & 2x+1 & 2x \\ x+1 & x+2 & 2x+1 & 2 & 0 & x & 1 & 2x & 2x+2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & x^0 & x^1 & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ x^0 & x^4 & x^7 & x^3 & x^5 & 0 & x^2 & x^1 & x^6 \\ x^1 & x^7 & x^5 & x^0 & x^4 & x^6 & 0 & x^3 & x^2 \\ x^2 & x^3 & x^0 & x^6 & x^1 & x^5 & x^7 & 0 & x^4 \\ x^3 & x^5 & x^4 & x^1 & x^7 & x^2 & x^6 & x^0 & 0 \\ x^4 & 0 & x^6 & x^5 & x^2 & x^0 & x^3 & x^7 & x^1 \\ x^5 & x^2 & 0 & x^7 & x^6 & x^3 & x^1 & x^4 & x^0 \\ x^6 & x^1 & x^3 & 0 & x^0 & x^7 & x^4 & x^2 & x^5 \\ x^7 & x^6 & x^2 & x^4 & 0 & x^1 & x^0 & x^5 & x^3 \end{bmatrix}$$

Then

$$A_1=\begin{bmatrix} 1&2&0&2&0&2&0&2&0 \\ 2&2&0&0&0&1&2&0&2 \\ 0&0&0&2&2&2&1&0&2 \\ 2&0&2&2&0&0&0&1&2 \\ 0&0&2&0&0&2&2&2&1 \\ 2&1&2&0&2&2&0&0&0 \\ 0&2&1&0&2&0&0&2&2 \\ 2&0&0&1&2&0&2&2&0 \\ 0&2&2&2&1&0&2&0&0 \end{bmatrix} \quad A_2=\begin{bmatrix} 1&0&2&0&2&0&2&0&2 \\ 0&0&2&2&2&1&0&2&0 \\ 2&2&2&0&0&0&1&2&0 \\ 0&2&0&0&2&2&2&1&0 \\ 2&2&0&2&2&0&0&0&1 \\ 0&1&0&2&0&0&2&2&2 \\ 2&0&1&2&0&2&2&0&0 \\ 0&2&2&1&0&2&0&0&2 \\ 2&0&0&0&1&2&0&2&2 \end{bmatrix}$$

Hence the *PB* array is

$$A=[A_1 \;\vdots\; A_2].$$

SUMMARY

A method of construction of three-symbol partially balanced (PB) arrays of strength two is put forward. Such arrays can be constructed for v factors in $2v$ assemblies whenever v is a prime ($\neq 2$) or an odd prime power. It may be mentioned here that these PB arrays have already been used by Gupta [4] and Gupta and Dey [5] to construct main effect plans for 3^n factorials and second order rotatable designs of six levels, respectively.

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